Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 6954802, 7 pages
https://doi.org/10.1155/2022/6954802

Research Article

On Regularity Criteria via Pressure for the 3D MHD Equations in a Half Space

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Received 5 March 2022; Accepted 30 June 2022; Published 26 July 2022

Academic Editor: Ricardo Weder

In this paper, we give regularity criteria in terms of the magnetic pressure in Lorentz spaces.

1. Introduction

We study the regularity issues for suitable weak solutions (u, b, π): $Q_T \longrightarrow \mathbb{R}^3_+ \times \mathbb{R}^3_+ \times \mathbb{R}$ of 3D incompressible magnetohydrodynamic (MHD) equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b = -\nabla\left(p + \dfrac{|b^2|}{2}\right), \\ \dfrac{\partial b}{\partial t} - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0 \text{ on } \Omega_T \coloneqq \mathbb{R}^3_{ss+} \times (0, T), \\ \text{div } u = \text{div } b = 0, \\ u(x, 0) = u_0(x) \text{ and } b(x, 0) = b_0(x). \end{cases} \tag{1}$$

Here, u is the fluid flow, b is the magnetic vector field, and $\pi = p + (|b|^2/2)$ is the total scalar pressure. We consider equation (1) with boundary conditions defined as follows: either

$$(B1)\ u = 0, b \cdot n = 0, (\nabla \times b) \times n = 0, \tag{2}$$

or

$$(B2)\ u \cdot n = 0, (\nabla \times u) \times n = 0, b \cdot n = 0, (\nabla \times b) \times n = 0, \tag{3}$$

where n is the outward unit normal vector along boundary $\partial\mathbb{R}^3_+$.

In pioneering works [1, 2], it has been shown that global-in time weak solutions to the MHD equations exist in finite energy space and strong solutions can exist locally-in time. In other words, the weak solutions exist globally in time; however, if a weak solution (u, b) are furthermore in $L^\infty(0, T; H^1(\Omega))$, they become regular. The regular solution means that $\|u\|_{L^\infty(Q_T)} + \|b\|_{L^\infty(Q_T)} < \infty$. The uniqueness and regularity of weak solutions to (1) have been left the question open. The authors in [3], very recently, the existence of global weak solutions to the 3D MHD equations via new energy control methods are inspired of a recent work [4]. On the other hand, for nonuniqueness, the author in [5] nonunique weak solutions in Leray-Hopf class are constructed for (1) in a whole space based on appreciated convex integration framework developed in a recent work of Buckmaster and Vicol [6]. In the regularity theory of weak solutions to fluid equations, the role of the pressure is very important (see [7, 8]); in particular, it is a more important issue for the boundary value problems. In present paper, we obtain the scaling invariant regularity criterion by focusing on the (magnetic) pressure function.

Note that equation (1) has the following scale:

$$\begin{aligned} u_\lambda &= \lambda u(\lambda x, \lambda^2 t), \\ b_\lambda &= \lambda b(\lambda x, \lambda^2 t), \\ \pi_\lambda &= \lambda^2 \pi(\lambda x, \lambda^2 t), \\ \lambda &> 0. \end{aligned} \tag{4}$$

For the regularity conditions in Sobolev space, the results in terms of magnetic pressure and the gradient of magnetic pressure for (1) in $\mathbb{R}^3$ were obtained by Zhou [9] with some magnetic field condition (see also [10–15]). After that, Duan [16] showed $\pi \in L^p(0, T; L^q(\mathbb{R}^3))$ with $2/p + 3/q = 2, q > 3/2$ or $\nabla\pi \in L^p(0, T; L^q(\mathbb{R}^3))$ with $2/p + 3/q = 3, q > 1$.

On the other hand, for the regularity criteria in Lorentz space, He and Wang [17] proved that a weak solution (u, b) for 3D MHD equations becomes regular under the scaling invariant conditions, the so-called Serrin's conditions, $u \in L^{q,\infty}(0, T; L^{p,\infty}(\mathbb{R}^3))$ with $3/p + 2/q \le 1$ and $p > 3$ or $\nabla u \in L^{q,\infty}(0, T; L^{p,\infty}(\mathbb{R}^3))$ with $3/p + 2/q \le 2$ and $p > 3/2$ (compared to [7, 18–26] for Navier-Stokes equations). In particular, for the magnetic pressure, Suzuki [24, 25] proved the regularity criteria to the Navier-Stokes equations in the Lorentz space under the assumption for the pressure via the truncation method introduced by [27]; namely, if $\pi \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3))$ and $\|\pi\|_{L^{p,\infty}(0,T;L^{q,\infty}(\mathbb{R}^3))} \le \varepsilon$ with $2/p + 3/q = 2, 5/2 < q < \infty$ or $\nabla\pi \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3))$ and $\|\nabla\pi\|_{L^{p,\infty}(0,T;L^{q,\infty}(\mathbb{R}^3))} \le \varepsilon$ with $2/p + 3/q = 3, 5/3 \le q < 3$, (u, b) is regular.

In this respect, the main results in the present paper are stated as follows.

Theorem 1. *Suppose that (u, b, π) is a weak solution to (1) with the divergence-free initial data $u_0, b_0 \in H^2(\mathbb{R}^3_+) \cap W^{1,q}(\mathbb{R}^3_+)$, $q > 3$. Then, there exists a constant $\varepsilon > 0$ such that $u(x, t)$ is a regular solution on $(0, T]$ provided that one of the following two conditions holds:*

(A) Under the boundary condition (B2), $\pi \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3_+))$ and

$$\|\pi\|_{L^{p,\infty}\left(0,T;L^{q,\infty}\left(\mathbb{R}^3_+\right)\right)} \le \varepsilon, \text{ with } 2/p + 3/q = 2, 3/2 < q < \infty \tag{5}$$

(B) Under the boundary conditions (B1) or (B2),

$$\nabla\pi \in L^p\left(0, T; L^{q,\infty}\left(\mathbb{R}^3_+\right)\right) \text{ with } 2/p + 3/q = 3, 1 < q < \infty \tag{6}$$

Remark 2. Theorem 1 is worth to extend the results of Theorem 4.1 in [28] to the Lorentz space in $\mathbb{R}^3_+$. The result of Theorem 1 is naturally expandable for the n-dimensional half space with aid of Sobolev embedding and Calderon-Zygmund inequalities.

Remark 3. Unlike the results in [29], Theorem 1 is valuable as a result of considering boundary conditions.

Remark 4. In light of the approach in [30], under the boundary conditions (B2), we can show the regularity condition of weak solutions to (1) with one component of the gradient of pressure, namely,

$$\partial_{x_3}\pi \in L^p\left(0, T; L^{q,\infty}\left(\mathbb{R}^3_+\right)\right) \text{ with } 2/p + 3/q \le 2, 1 < q < \infty. \tag{7}$$

Remark 5. In part (B) of Theorem 1, unfortunately, it does not obtain a similar result as (A) due to the difficulty of controlling the pressure function from the complexity of mixed term for w^+ and w^- (see Remark 11).

For the Navier-Stokes equations with boundary data (B1) or (B2), Theorem 1 immediately implies.

Corollary 6. *Suppose that (u, p) is a weak solution to the Navier-Stokes equations. Then, there exists a constant $\varepsilon > 0$ such that $u(x, t)$ is a regular solution on $(0, T]$ provided that one of the following two conditions holds:*

(A) Under the boundary condition (B2), $\pi \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3_+))$ and

$$\|\pi\|_{L^{p,\infty}\left(0,T;L^{q,\infty}\left(\mathbb{R}^3_+\right)\right)} \le \varepsilon, \text{ with } 2/p + 3/q = 2, 3/2 < q < \infty \tag{8}$$

(B) Under the boundary conditions (B1) or (B2), $\nabla\pi \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3_+))$ and

$$\|\nabla\pi\|_{L^{p,\infty}\left(0,T;L^{q,\infty}\left(\mathbb{R}^3_+\right)\right)} \le \varepsilon, \text{ with } 2/p + 3/q = 3, 1 < q < \infty \tag{9}$$

The proof of Corollary 6 is same to that in [31] and thus it is omitted.

2. Notations and Some Auxiliary Lemmas

For $p \in [1,\infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions $f(x, t)$ on the interval $(0, T)$ with values in X and $\|f(\cdot, t)\|_X$ belonging to $L^p(0, T)$. The space $W^{k,2}(\Omega)$ is denoted the standard Sobolev space. For a function $f(x, t)$, $\Omega \subset \mathbb{R}^3$, we denote $\|f\|_{L^{p,q}_{x,t}(\Omega\times I)} = \|f\|_{L^q_t(I;L^p_x(\Omega))} = \|\|f\|_{L^p_x(\Omega)}\|_{L^q_t(I)}$. C is a generic constant.

We recall first the definition of weak solutions.

Definition 7 (weak solutions). The vector-valued function (u, b) is called a weak solution of (1) on $(0, T) \times \mathbb{R}^3_+$ if it satisfies the following conditions:

(1) $(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3_+)) \cap L^2(0, T; H^1(\mathbb{R}^3_+))$

(2) div $u =$ div $b = 0$ in the sense of distribution

(3) For any function $\psi(t, x) \in C_0^\infty((0, T) \times \mathbb{R}^3_+)$ with div $\psi = 0$, there hold

$$\int_0^T\int_{\mathbb{R}^3_+}\{u\cdot\psi_t - \nabla u\cdot\nabla\psi + \nabla\psi : (u\otimes u - b\otimes b)\}dxdt = 0,$$
$$\int_0^T\int_{\mathbb{R}^3_+}\{b\cdot\psi_t - \nabla b\cdot\nabla\psi + \nabla\psi : (u\otimes b - b\otimes u)\}dxdt = 0 \tag{10}$$

Next, we give some basic facts. For $p, q \in [1,\infty]$, we define

$$\|f\|_{L^{p,q}(\mathbb{R}^3_+)} = \begin{cases} \left(p\int_0^\infty \alpha^q |\{x\in\Omega : |f(x)|>\alpha\}|^{q/p}\frac{d\alpha}{\alpha}\right)^{1/q}, & q<\infty, \\ \sup_{\alpha>0}\alpha\left|\{x\in\mathbb{R}^3_+ : |f(x)|>\alpha\}\right|^{1/p}, & q=\infty. \end{cases} \tag{11}$$

And thus,

$$L^{p,q}(\mathbb{R}^3_+) = \left\{f : f \text{ is a measurable function on } \mathbb{R}^3_+ \text{ and } \|f\|_{L^{p,q}(\mathbb{R}^3_+)}<\infty\right\}. \tag{12}$$

Followed in [32], the Lorentz space $L^{p,q}(\mathbb{R}^3_+)$ may be defined by real interpolation methods

$$L^{p,q}(\mathbb{R}^3_+) = \left(L^{p_1}(\mathbb{R}^3_+), L^{p_2}(\mathbb{R}^3_+)\right)_{\alpha,q}, \tag{13}$$

with

$$\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}, \quad 1\le p_1 < p < p_2 \le \infty, \tag{14}$$

that is,

$$L^{2p/(p-1),2}(\mathbb{R}^3_+) = \left(L^2(\mathbb{R}^3_+), L^6(\mathbb{R}^3_+)\right)_{3/2p,2}. \tag{15}$$

We list some lemmas for our analysis.

Lemma 8. ([33]). *Assume $1\le p_1,\ p_2\le\infty,\ 1\le q_1,\ q_2\le\infty$, $u\in L^{p_1,q_1}(\Omega)$, and $v\in L^{p_2,q_2}(\Omega)$. Then, $uv\in L^{p_3,q_3}(\Omega)$ with $1/p_3 = (1/p_1)+(1/p_2)$ and $1/q_3 \le (1/q_1)+(1/q_2)$, and the inequality*

$$\|uv\|_{L^{p_3,q_3}(\Omega)} \le C\|u\|_{L^{p_1,q_1}(\Omega)}\|v\|_{L^{p_2,q_2}(\Omega)} \tag{16}$$

is valid.

Lemma 9 ([20, 34, 35]). *Let $T>0$ and $\phi\in L_{loc}([0,T))$ be nonnegative function. Assume further that*

$$\phi(t) \le C_0 + C_1\int_0^t \mu(s)\phi(s)\,ds + \kappa\int_0^t \lambda(s)^{1-\varepsilon}\phi(s)^{1+A(\varepsilon)}\,ds, \quad \forall 0<\varepsilon<\varepsilon_0, \tag{17}$$

where $\kappa, \varepsilon_0>0$ are constants, $\mu\in L^1(0,T)$, and $A(\varepsilon)>0$ satisfies $\lim_{\varepsilon\longrightarrow 0}A(\varepsilon)/\varepsilon = c_0>0$. Then ϕ is bounded on $[0,T]$ if $\|\lambda\|_{L^{1,\infty}(0,T)} < c_0^{-1}\kappa^{-1}$.

Lemma 10 ([31]). *Assume that the pair (p,q) satisfies $(2/p)+(3/q)=a$ with $a, q\ge 1$ and $p>0$. Then, for every $\kappa\in[0,1]$ and given $b, c_0\ge 1$, there exist $p_\kappa>0$ and* $\min\{q,b\}\le q_\kappa\le\max\{q,b\}$ *such that*

$$\begin{cases} \dfrac{2}{p_\kappa} + \dfrac{3}{q_\kappa} = a, \\ \dfrac{p_\kappa}{q_\kappa} = \dfrac{p(1-\kappa)}{q} + \dfrac{c_0\kappa}{b}. \end{cases} \tag{18}$$

3. Proof of Theorems: Half Space Case

Proof of Theorem 1. We rewrite equation (1) with $w^+ = u + b$ and $w^- = u - b$:

$$\begin{cases} w^+_t - \Delta w^+ + (w^-\cdot\nabla)w^+ = \nabla\pi, \operatorname{div} w^+ = 0, \\ w^-_t - \Delta w^- + (w^+\cdot\nabla)w^- = \nabla\pi, \operatorname{div} w^- = 0, \\ w^+(x,0) = w^+_0 \text{ and } w^-(x,0) = w^-_0. \end{cases} \tag{19}$$

Part (A): multiplying both side of (19) by $w^+|w^+|^2$, integrating by parts with the divergence-free condition, we conclude that

$$\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3_+}|w^+|^4dx + \int_{\mathbb{R}^3_+}|\nabla w^+|^2|w^+|^2dx + \frac{1}{2}\int_{\mathbb{R}^3_+}|\nabla|w^+|^2|^2dx = -\int_{\mathbb{R}^3_+}w^+\cdot\nabla\pi|w^+|^2dx = I. \tag{20}$$

Using the integration by parts and Hölder inequality, we have

$$I = \int_{\mathbb{R}^3_+}\pi w^+\cdot\nabla|w^+|^2dx \le C\int_{\mathbb{R}^3_+}\pi^2|w^+|^2dx + \frac{1}{8}\int_{\mathbb{R}^3_+}|\nabla w^+|^2|w^+|^2dx. \tag{21}$$

By means of the Hölder, interpolation, and Sobolev embedding inequalities in the Lorentz spaces,

$$\left\| |w^+|^2 \right\|_{L^{2q/(q-1),2}(\mathbb{R}^3_+)} \le \left\| |w^+|^2 \right\|_{L^2(\mathbb{R}^3_+)}^{1-(3/2q)} \left\| |w^+|^2 \right\|_{L^{6,2}(\mathbb{R}^3_+)}^{3/q} \le C \left\| |w^+|^2 \right\|_{L^2(\mathbb{R}^3_+)}^{1-(3/2q)} \left\| \nabla |w^+|^2 \right\|_{L^2(\mathbb{R}^3_+)}^{3/2q}. \quad (22)$$

On the other hand, for a magnetic pressure, following the approach of Theorem 2.1 in [36], it is easy to check that

$$\|\pi\|_{L^p(\mathbb{R}^3_+)} \le C\left(\|w^+\|^2_{L^{2p}(\mathbb{R}^3_+))} + \|w^-\|^2_{L^{2p}(\mathbb{R}^3_+))} \right), 1 < p < \infty. \quad (23)$$

With the help of the Hölder inequality with estimates (22) and (23), we infer that

$$\begin{aligned} \int_{\mathbb{R}^3_+} \pi^2 |w^+|^2 dx &\le \|\pi\|_{L^{q,\infty}(\mathbb{R}^3_+)} \|\pi\|_{L^{2q/(q-1),2}(\mathbb{R}^3_+)} \left\| |w^+|^2 \right\|_{L^{2q/(q-1),2}(\mathbb{R}^3_+)} \\ &\le C\|\pi\|_{L^{q,\infty}(\mathbb{R}^3_+)} \left(\left\| |w^+|^2 \right\|_{L^{2q/(q-1),2}(\mathbb{R}^3_+)} + \left\| |w^-|^2 \right\|_{L^{2q/(q-1),2}(\mathbb{R}^3_+)} \right) \left\| |w^+|^2 \right\|_{L^{2q/(q-1),2}(\mathbb{R}^3_+)} \\ &\le C\|\pi\|^{2q/(2q-3)}_{L^{q,\infty}(\mathbb{R}^3_+)} \left(\left\| |w^+|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} + \left\| |w^-|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} \right) \\ &\quad + \frac{1}{8}\left(\| |w^+| |\nabla w^+| \|^2_{L^2(\mathbb{R}^3_+)} + \| |w^-| |\nabla w^-| \|^2_{L^2(\mathbb{R}^3_+)} \right). \end{aligned} \quad (24)$$

And thus, estimate (20) becomes

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3_+} |w^+|^4 dx + \int_{\mathbb{R}^3_+} |\nabla w^+|^2 |w^+|^2 dx \\ &\le C\|\pi\|^{2q/(2q-3)}_{L^{q,\infty}(\mathbb{R}^3_+)} \left(\left\| |w^+|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} + \left\| |w^-|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} \right) \\ &\quad + \frac{1}{8}\left(\| |w^+| |\nabla w^+| \|^2_{L^2(\mathbb{R}^3_+)} + \| |w^-| |\nabla w^-| \|^2_{L^2(\mathbb{R}^3_+)} \right). \end{aligned} \quad (25)$$

Similarly, we have

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3_+} |w^-|^4 dx + \int_{\mathbb{R}^3_+} |\nabla w^-|^2 |w^-|^2 dx \\ &\le C\|\pi\|^{2q/(2q-3)}_{L^{q,\infty}(\mathbb{R}^3_+)} \left(\left\| |w^+|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} + \left\| |w^-|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} \right) \\ &\quad + \frac{1}{8}\left(\| |w^+| |\nabla w^+| \|^2_{L^2(\mathbb{R}^3_+)} + \| |w^-| |\nabla w^-| \|^2_{L^2(\mathbb{R}^3_+)} \right). \end{aligned} \quad (26)$$

Summing (39) and (40), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3_+} \left(|w^+|^4 + |w^-|^4 \right) dx + \frac{1}{2}\int_{\mathbb{R}^3_+} \left(|\nabla w^+|^2 |w^+|^2 + |\nabla w^-|^2 |w^-|^2 \right) dx \\ &\le C\|\pi\|^{2q/(2q-3)}_{L^{q,\infty}(\mathbb{R}^3_+)} \left(\left\| |w^+|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} + \left\| |w^-|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} \right). \end{aligned} \quad (27)$$

Let $\mathfrak{N}(t) := \|w^+\|^4_{L^4(\mathbb{R}^3_+)} + \|w^-\|^4_{L^4(\mathbb{R}^3_+)}$, and thus, (27) becomes

$$\frac{d}{dt}\mathfrak{N}(t) \le C\|\pi\|^p_{L^{q,\infty}(\mathbb{R}^3_+)} \mathfrak{N}(t), p = \frac{2q}{2q-3}. \quad (28)$$

Applying Lemma 10 (with $a = b = 2, c_0 = 4$), we have

$$\begin{aligned} \|\pi\|^{p_\kappa}_{L^{q_\kappa,\infty}(\mathbb{R}^3_+)} &\le \|\pi\|^{p(1-\kappa)}_{L^{q,\infty}(\mathbb{R}^3_+)} \|\pi\|^{4\kappa}_{L^{2,\infty}} \le C\|\pi\|^{p(1-\kappa)}_{L^{q,\infty}(\mathbb{R}^3_+)} \|\pi\|^{4\kappa}_{L^2(\mathbb{R}^3_+)} \\ &\le C\|\pi\|^{p(1-\kappa)}_{L^{q,\infty}(\mathbb{R}^3_+)} \left\| |u|^2 \right\|^{4\kappa}_{L^2(\mathbb{R}^3_+)}, \end{aligned} \quad (29)$$

where we use the following estimate in [37]:

$$\|f\|_{L^{p,q_2}(\mathbb{R}^3_+)} \le \left(\frac{q_1}{p}\right)^{(1/q_1)-(1/q_2)} \cdot \|f\|_{L^{p,q_1}(\mathbb{R}^3_+)}, 1 \le p \le \infty, 1 \le q_1 < q_2 \le \infty. \quad (30)$$

Since the pair (p_κ, q_κ) also meets $2/p_\kappa + 3/q_\kappa = 2$, using estimate (29), (28) becomes

$$\frac{d}{dt}\mathfrak{N}(t) \le C\|\pi\|^{p_\kappa}_{L^{q_\kappa,\infty}(\mathbb{R}^3_+)} \left\| |u|^2 \right\|^2_{L^2(\mathbb{R}^3_+)} \le C\|\pi\|^{p(1-\kappa)}_{L^{q,\infty}(\mathbb{R}^3_+)} Y(t)^{1+2\kappa}. \quad (31)$$

And then integrating with respect to time, we get

$$\mathfrak{N}(t) \le C\mathfrak{N}(0) + C\int_0^t \|\pi\|^{p(1-\kappa)}_{L^{q,\infty}(\mathbb{R}^3_+)} \mathfrak{N}(t)^{1+2\kappa} ds, \quad (32)$$

or equivalently,

$$\begin{aligned} &\|w^+(t)\|^4_{L^4(\mathbb{R}^3_+)} + \|w^-(t)\|^4_{L^4(\mathbb{R}^3_+)} \\ &\le C\|w_0^+\|^4_{L^4(\mathbb{R}^3_+)} + \|w_0^-\|^4_{L^4(\mathbb{R}^3_+)} + C\int_0^t \|\pi\|^{p(1-\kappa)}_{L^{q,\infty}(\mathbb{R}^3_+)} \\ &\quad \cdot \left(\|w^+\|^4_{L^4(\mathbb{R}^3_+)} + \|w^-\|^{4(1+2)\kappa}_{L^4(\mathbb{R}^3_+)} \right) ds. \end{aligned} \quad (33)$$

Due to Lemma 9, we complete the Proof of Theorem 1 under the assumption (A) in Theorem 1.

Part (B): for this, we use the argument in [16], which seems like simple method to deal with the pressure term.

Multiplying both side of (19) by $w^+|w^+|^{3r-4}$, we conclude that for $r \geq 1$,

$$\begin{aligned}&\frac{1}{3r-2}\frac{d}{dt}\int_{\mathbb{R}^3_+}|w^+|^{3r-2}dx + \frac{4(3r-4)}{(3r-2)^2}\int_{\mathbb{R}^3_+}\left|\nabla|w^+|^{(3r-2)/2}\right|^2 dx\\&\quad+\int_{\mathbb{R}^3_+}|\nabla w^+|^2|w^+|^{3r-4}dx\\&\quad= -\int_{\mathbb{R}^3_+}\nabla\pi\cdot w^+|w^+|^{3r-4}\,dx := II.\end{aligned} \tag{34}$$

On the other hand,

$$\begin{aligned}II &\leq (3r-4)\int_{\mathbb{R}^3_+}|\pi||\nabla|w^+||w^+|^{3r-4}dx\\&\leq \frac{2(3r-4)}{(3r-2)}\left(\int_{\mathbb{R}^3_+}|\pi|^2|w^+|^{3r-4}\,dx\right)^{1/2}\\&\quad\cdot\left(\int_{\mathbb{R}^3_+}|\nabla|w^+|^{(3r-2)/2}|dx\right)^{1/2}.\end{aligned} \tag{35}$$

Note that $0 \leq I \leq a$ and $0 \leq I \leq b$; then, $I \leq \sqrt{ab}$. Combining (34) and (35), we get

$$\begin{aligned}II &\leq C\left(\int_{\mathbb{R}^3_+}\nabla\pi\cdot w^+|w^+|^{3r-4}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3_+}|p|^2\bigg\|^{3r-3}dx\right)^{1/4}\\&\quad\cdot\left(\int_{\mathbb{R}^3_+}\left|\nabla|w^+|^{(3r-2)/2}\right|dx\right)^{1/4}\\&\leq C\left(\int_{\mathbb{R}^3_+}\left(|\nabla\pi|\left(|w^+|^2+|w^-|^2\,dx\right)^{(3r-3)/2}\right)\right)^{2/3}\\&\quad\cdot\left(\int_{\mathbb{R}^3_+}\left(|\pi|\left(|w^+|^2+|w^-|^2\,dx\right)^{(3r-4)/2}\right)\right)^{1/4}\\&\quad+\frac{3r-4}{(3r-2)^2}\left(\int_{\mathbb{R}^3_+}\left|\nabla|w^+|^{(3r-2)/2}\right|^2dx\right).\end{aligned} \tag{36}$$

Due to

$$\begin{aligned}&\int_{\mathbb{R}^3_+}|\pi|\left(|w^+|^2+|w^-|^{(3r-4)/2}\right)dx \leq \|\pi\|^2_{L^{3r/2}}\||w^+|+|w^-|\|^{(3r-4)/2}_{L^{3r/2}},\\&\int_{\mathbb{R}^3_+}|\nabla\pi|\left(|w^+|^2+|w^-|^{(3r-3)/2}\right)dx \leq \|\nabla\pi\|_{L^r}\||w^+|+|w^-|\|^{(3r-3)/2}_{L^{3r/2}},\end{aligned} \tag{37}$$

we can know that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3_+}|w^+|^{3r-2}dx+\int_{\mathbb{R}^3_+}\left|\nabla|w^+|^{(3r-2)/2}\right|^2dx+\int_{\mathbb{R}^3_+}|\nabla w^+|^2|w^+|^{3r-4}dx\\&\quad\leq\|\nabla\pi\|^{2/3}_{L^r}\||w^+|+|w^-|\|^{(3r-2)/2}_{L^{3r/2}}.\end{aligned} \tag{38}$$

In a similar fashion, if you do it for equation (20), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3_+}|w^-|^{3r-2}dx+\int_{\mathbb{R}^3_+}\left|\nabla|w^-|^{(3r-2)/2}\right|^2dx+\int_{\mathbb{R}^3_+}|\nabla w^+|^2|w^+|^{3r-4}dx\\&\quad\leq\|\nabla\pi\|^{2/3}_{L^r}\||w^+|+|w^-|\|^{(3r-2)/2}_{L^{3r/2}}.\end{aligned} \tag{39}$$

After summing up (38) and (39), using the Sobolev embedding and Young's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3_+}\left(|w^+|^{3r-2}+|w^-|^{3r-2}\right)dx\\&\quad+\int_{\mathbb{R}^3_+}\left(\left|\nabla|w^+|^{(3r-2)/2}\right|^2+\left|\nabla|w^-|^{(3r-2)/2}\right|^2\right)dx\\&\quad+\int_{\mathbb{R}^3_+}|\nabla w^+|^2|w^+|^{3r-4}dx+\int_{\mathbb{R}^3_+}|\nabla w^-|^2|w^-|^{3r-4}dx\\&\quad\leq C\|\nabla\pi\|^{2/3}_{L^{r,\infty}}\left(\left\||w^+|^{(3r-2)/2}\right\|^2_{L^{6r/(3r-2)},1}+\left\||w^-|^{(3r-2)/2}\right\|^2_{L^{6r/(3r-2)},1}\right)\\&\quad\leq C\|\nabla\pi\|^{2/3}_{L^{r,\infty}}\left(\left\||w^+|^{(3r-2)/2}\right\|^{(1-(1/r))}_{L^2}\left\|\nabla|w^+|^{(3r-2)/2}\right\|^{2/r}_{L^2}\right.\\&\quad\left.+\left\||w^-|^{(3r-2)/2}\right\|^{(1-(1/r))}_{L^2}\left\|\nabla|w^-|^{(3r-2)/2}\right\|^{2/r}_{L^2}\right)\\&\quad\leq C\|\nabla\pi\|^{2r/(3r-3)}_{L^{r,\infty}}\left(\left\||w^+|^{(3r-2)/2}+|w^-|^{(3r-2)/2}\right\|^2_{L^2}\right)\\&\quad+\frac{1}{8}\int_{\mathbb{R}^3_+}\left(\left|\nabla|w^+|^{(3r-2)/2}\right|^2+\left|\nabla|w^-|^{(3r-2)/2}\right|^2\right)dx\\&\quad\leq C\|\nabla\pi\|^{2r/(3r-3)}_{L^{r,\infty}}\left(\|w^+\|^{3r-2}_{L^{3r-2}(\mathbb{R}^3_+)}+\|w^-\|^{3r-2}_{L^{3r-2}}\right)\\&\quad+\frac{1}{8}\int_{\mathbb{R}^3_+}\left(\left|\nabla|w^+|^{(3r-2)/2}\right|^2+\left|\nabla|w^-|^{(3r-2)/2}\right|^2\right)dx.\end{aligned} \tag{40}$$

Let $\mathfrak{N}(t) := \|w^+\|^{3r-2}_{L^{3r-2}(\mathbb{R}^3_+)}+\|w^-\|^{3r-2}_{L^{3r-2}(\mathbb{R}^3_+)}$, and then, (40) becomes

$$\mathfrak{N}(t) \leq C\|\nabla\pi\|^{2q/(3q-3)}_{L^{q,\infty}(\mathbb{R}^3_+)}\mathfrak{N}(t). \tag{41}$$

As the previous way, it allows us to finish the Proof of Theorem 1. □

Remark 11. In part (B) of Theorem 1, adding the following conditions

$$|u||\nabla u|, |b||\nabla b| \in L^2\left(0, T; L^2\left(\mathbb{R}^3_+\right)\right), \tag{42}$$

we also can obtain $\nabla\pi \in L^{p,\infty}(0, T; L^{q,\infty}(\mathbb{R}^3_+))$ and

$$\|\nabla\pi\|_{L^{p,\infty}\left(0,T;L^{q,\infty}\left(\mathbb{R}^3_+\right)\right)} \le \varepsilon, \text{ with } 2/p + 3/q = 3, 1 < q < \infty \tag{43}$$

(see [31] for a detailed proof). Condition (42) is too strong because it is regular condition of weak solutions to (1) (see, e.g., Lemma 7 in [38] or [39]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflict of interest.

Acknowledgments

Jae-Myoung Kim was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521).

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